Multithreshold Multipartite Graphs

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Abstract

We call a graph G a *k-threshold graph* if there are k distinct real numbers $\theta_1, \theta_2, \ldots, \theta_k$ and a mapping $r : V(G) \to \mathbb{R}$ such that for any two vertices $u, v \in V(G)$, we have that $uv \in E(G)$ if and only if there are odd numbers θ_i such that $\theta_i \leq r(u) + r(v)$. The least integer k such that G is a k-threshold graph is called a *threshold number* of G, and denoted by $\Theta(G)$. The well-known family of threshold graphs is a set of graphs G with $\Theta(G) \leq 1$. Jamison and Sprague in [Multithreshold graphs, *J. Graph Theory*, 94(4): 518-530, 2020] introduced the concept of k-threshold graph, and proved that $\Theta(G)$ exists for every graph G. They further obtained a number of interesting results on $\Theta(G)$. In addition, they also proposed several unsolved problems and conjectures, including the following two.

- **Problem**: Determine the exact threshold numbers of the complete multipartite graphs.
- **Conjecture**: For all even $n \geq 2$, there is a graph G with $\Theta(G) = n$ and $\Theta(G^c) = n+1$. This is equivalent to that for all odd $n \geq 3$, there is a graph G with $\Theta(G) = n$ and $\Theta(G^c) = n - 1$, where G^c is the complement of G.

In this short paper, we give a partial solution of the problem and confirm the conjecture.

Keywords: Complete multipartite graphs; Threshold graphs; Vertex ranks; Graph representation; Threshold numbers

1 Introduction

We will primarily use the notation and terminologies from West [8]. In this paper, we consider simple graphs, i.e., finite, undirected, and no loops or multiple edges. Let G be a graph and G^c be the complement of G. Denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For an edge set $F \subseteq E(G)$, let $G - F$ be the graph obtained by removing all edges in F from G; for an edge set $F \subseteq E(G^c)$, let $G + F$ denote the graph obtained by adding all edges in F to G. The neighborhood of vertex $v \in V(G)$, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v.

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A graph G is said to be a *threshold graph* if there is an assignment $r : V(G) \to \mathbb{R}$ of real ranks to the vertices such that $uv \in E(G)$ if and only if $r(u) + r(v) \geq 0$ for any two vertices $u, v \in V(G)$. The family of threshold graphs, which represents a well-studied class of graphs from numerous directions, was introduced by Chvátal and Hammer [1] in 1977. Since then, this family of graphs has been extensively studied and a great deal of intriguing results are acquired. (see [7, 6, 4, 3, 2].)

Jamison and Sprague [5] recently introduced multithresholds as follows. A graph G is called a *k-threshold graph* if there exist k thresholds $\theta_1 < \theta_2 < \cdots < \theta_k$ and an assignment $r : V(G) \to \mathbb{R}$ of real ranks to the vertices such that, for any two vertices $u, v \in V(G)$, $uv \in E(G)$ if and only if the number of θ_i with $\theta_i \leq r(u) + r(v)$ is odd. The above assignment r and the choice of k thresholds such that G is k-threshold graph is called a *k-threshold representation* of G. The *threshold number* of a graph G, denoted by $\Theta(G)$, is the smallest k such that G has a k-threshold representation. It is readily seen that any k-threshold representation of G can be converted to a $(k+1)$-threshold representation of G as long as a threshold $\theta_{k+1} > \theta_k$ is added without changing the rank of vertices. Therefore, if $\Theta(G) = k$ then G is an ℓ-threshold graph for any integer $\ell \geq k$. In addition, we also note that $\Theta(H) \leq \Theta(G)$ if H is an induced subgraph of G.

In the same paper, Jamison and Sprague showed a number of interesting results as follows. For any graph G of order n, $\Theta(G)$ exists and $\Theta(G) \leq \frac{n(n-1)}{2}$. For any path P with at least 4 vertices, $\Theta(P) = 2$. More generally, for any caterpillar T, $\Theta(T) \leq 2$. For any two distinct vertices v, w of a graph G, if $vw \in E(G)$ then $\Theta(G - vw) \leq \Theta(G) + 2$, and if $vw \in E(G^c)$ then $\Theta(G + vw) \leq \Theta(G) + 2$. Furthermore, they gave a lower bound of the threshold numbers for the general graphs as well. The following result will be used in our proof.

Theorem 1.1 (Jamison and Sprague [5]). *The threshold number of G and its complement G^c differ by at most 1. More specifically, if $\Theta(G)$ is odd then $\Theta(G^c) \leq \Theta(G)$, and if $\Theta(G)$ is even then $\Theta(G^c) \geq \Theta(G)$.*

At the end of the paper, they put forward the following four problems and two conjectures, cited "which are immediately suggested for further work".

Problem 1. *Let Θ_n denote the largest threshold number among graphs of order n. How does Θ_n behave asymptotically?*

Problem 2. *Determine better bounds on the threshold numbers of the five special classes of graphs studied in the paper.*

Problem 3. *Determine the exact threshold numbers of the complete multipartite graphs.*

Problem 4. *Given an integer k, what is the computational complexity of determining whether the threshold number of a graph G is at most k?*

Conjecture 1. *For all even $n \geq 2$, there is a graph G with $\Theta(G) = n$ and $\Theta(G^c) = n+1$. This is equivalent to that for all odd $n \geq 3$, there is a graph G with $\Theta(G) = n$ and $\Theta(G^c) = n-1$.*

Conjecture 2. *The graphs achieving the maximum threshold number Θ_n have vertex degrees close to $n/2$.*

At the 2019 Spring Sectional AMS Meeting in Auburn, Jamison offered a \$50 bounty for an answer to Problem 3. In this paper, by showing the following Theorem 1.2, we solve the Problem 3 for complete multipartite graphs where each color classes is not too small, and determine the threshold number of their complements. The combination of the two results fully confirms Conjecture 1. Let $K_{n_1,n_2,\ldots,n_k}$ denote the complete k-partite graphs, where n_1, n_2, $\ldots$, n_k are the sizes of its color classes.

Theorem 1.2. *Let $n_1, \ldots, n_k$ be k positive integers. If $n_i \geq k+1$ for each $i \in \{1, 2, \ldots, k\}$, then $\Theta(K_{n_1,n_2,\ldots,n_k}) = 2k-2$ and $\Theta(K^c_{n_1,n_2,\ldots,n_k}) = 2k-1$.*

2 Proof of Theorem 1.2

We break Theorem 1.2 into four sub-theorems: Theorems 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4; and prove them one by one. In the remainder of this paper, we assume that X_1, X_2, $\ldots$, X_k are color classes of $K_{n_1,n_2,\ldots,n_k}$ with $|X_i| = n_i$ for each $i \in \{1, 2, \ldots, k\}$. Jamison and Sprague [5] proved that $\Theta(K_{n_1,n_2,\ldots,n_k}) \leq 2k$. We first improve their upper bound as below.

Theorem 2.1. $\Theta(K_{n_1,n_2,\ldots,n_k}) \leq 2k-2$.

Proof. Let $G := K_{n_1,n_2,\ldots,n_k}$. Define an assignment $r : V(G) \to \mathbb{R}$ such that the rank $r(v) = 3^i$ if $v \in X_i$ for some $i \in \{1, \ldots, k\}$. Let $\theta_i = 3^i$ and $\pi_i = 2 \cdot 3^i$ for $i \in \{2, 3, \ldots, k\}$ be thresholds. Clearly, every threshold is at least 9. We claim that these $2k-2$ thresholds and the assignment r give a $(2k-2)$-threshold representation of G. Let $u, v \in V(G)$ be any two distinct vertices.

Suppose that $uv \notin E(G)$, i.e., u and v are in the same color class, say X_i. We then have $r(u) = 3^i$ and $r(v) = 3^i$, which gives us that $r(u) + r(v) = 2 \cdot 3^i$. If $i = 1$ then $r(u) + r(v) = 6$. Hence, the number of thresholds less than or equal to $r(u)+r(v)$ is zero, and so is even. Suppose now that $i \geq 2$. Obviously, $\theta_h = 3^h \leq r(u) + r(v)$ if and only if $h \in \{2, \ldots, i\}$, and $\pi_h = 2 \cdot 3^h \leq r(u) + r(v)$ if and only if $h \in \{2, \ldots, i\}$. Hence, the number of thresholds less than or equal to $r(u) + r(v)$ is also even.

Suppose that $uv \in E(G)$. In this case, there are two distinct color classes X_i and X_j such that $u \in X_i$ and $v \in X_j$. We assume without loss of generality that $i < j$. Then $r(u) = 3^i$ and $r(v) = 3^j$, which gives us that $r(u)+r(v) < 2 \cdot 3^j$. Obviously, $\theta_h = 3^h \leq r(u)+r(v)$ if and only if

$h \in \{2, \ldots, j\}$ and $\pi_h = 2 \cdot 3^h \le r(u) + r(v)$ if and only if $h \in \{2, \ldots, j-1\}$. Hence, the number of thresholds less than or equal to $r(u) + r(v)$ is odd.

Therefore, G is $(2k-2)$-threshold graph, and so $\Theta(G) \le 2k-2$. $\square$

Remark 1. *Notice that when $n_1 = n_2 = \cdots = n_k = 1$, the graph $K_{1,1,\ldots,1}$ is the complete graph K_n on n vertices. Consequently, $\Theta(K_{1,1,\ldots,1}) = \Theta(K_n) = 1$. Jamison and Sprague [5] showed that $\Theta(K_{2,2,\ldots,2}) \le 3$. So, when $k \ge 3$, there are n_1, n_2, $\ldots$, n_k such that $\Theta(K_{n_1,n_2,\ldots,n_k}) < 2k-2$. It would be interesting to know the value of $\Theta(K_{3,3,\ldots,3})$.*

Theorem 2.2. *If $n_i \ge k+1$ for each $i \in \{1, \ldots, k\}$, then $\Theta(K_{n_1,n_2,\ldots,n_k}) \ge 2k-2$.*

Proof. Let $G := K_{n_1,n_2,\ldots,n_k}$ and assume $\Theta(G) = t$. Let $\theta_1 < \theta_2 < \cdots < \theta_t$ and an assignment $r : V(G) \to \mathbb{R}$ be a $t-$threshold representation of G. Next, we show $t \ge 2k-2$.

Claim 1. *For any two vertices $u, v \in V(G)$, if $r(u) = r(v)$ then $N(u)\backslash\{v\} = N(v)\backslash\{u\}$. It's equivalent to saying that if u, v are in different color classes then $r(u) \ne r(v)$.*

Proof. Suppose on the contrary that there is a vertex $w \in N(u)\backslash N(v)$ and $w \notin \{u, v\}$. Since $uw \in E(G)$, there are exactly odd number of thresholds θ_i such that $\theta_i \le r(u) + r(w)$. On the other hand, since $vw \notin E(G)$, there are exactly even number of thresholds θ_i such that $\theta_i \le r(v) + r(w)$. Consequently, $r(u) + r(w) \ne r(v) + r(w)$, giving a contradiction to $r(u) = r(v)$. $\square$

Claim 2. *Relabeling color classes X_1, X_2, $\ldots$, X_k if necessary, we may assume that there exist k pairs of vertices $u_i, v_i \in X_i$ with $i \in \{1, 2, \ldots, k\}$ such that* $\max\{r(u_i), r(v_i)\} < \min\{r(u_j), r(v_j)\}$ *whenever $i < j$, where $j \in \{1, 2, \ldots, k\}$.*

We apply Claim 2 to complete the proof of Theorem 2.2 before giving it a proof. Let k pairs of vertices u_i, v_i for $i \in \{1, 2, \ldots, k\}$ as stated in Claim 2. We assume without loss of generality that $r(u_i) \ge r(v_i)$ for each $i \in \{1, 2, \ldots, k\}$. Hence, for each $1 \le i \le k-1$, we have $r(v_i) \le r(u_i) < r(v_{i+1}) \le r(u_{i+1})$. Furthermore, we have

$$r(u_i) + r(v_i) < r(u_i) + r(v_{i+1}) < r(u_{i+1}) + r(v_{i+1}). \tag{1}$$

Since $u_iv_i \notin E(G)$ and $u_iv_{i+1} \in E(G)$, there are odd number of thresholds among θ_1, θ_2, $\ldots$, θ_t between $r(u_i) + r(v_i)$ and $r(u_i) + r(v_{i+1})$, and so there is at least one threshold θ_{ℓ_i} such that $r(u_i) + r(v_i) < \theta_{\ell_i} \le r(u_i) + r(v_{i+1})$. Similarly, we can show that there is at least one threshold θ_{m_i} such that $r(u_i) + r(v_{i+1}) < \theta_{m_i} \le r(u_{i+1}) + r(v_{i+1})$. By (1), we have $\theta_{\ell_i} < \theta_{m_i}$.

Note that in general we have the following chain of inequalities

$$r(v_1) \le r(u_1) < r(v_2) \le r(u_2) < \cdots < r(v_k) \le r(u_k).$$

So, we get that

$$\theta_{\ell_1} < \theta_{m_1} < \theta_{\ell_2} < \theta_{m_2} < \cdots < \theta_{\ell_{k-1}} < \theta_{m_{k-1}}.$$

Consequently, there are at least $2k-2$ thresholds, and so $\Theta(G) \geq 2k-2$. $\square$

Proof of Claim 2: For each $i \in \{1, 2, \ldots, k\}$, let $r(X_i)$ be the non-increasing list of ranks $r(x)$ of $x \in X_i$. We assume without loss of generality that among all k lists $r(X_1)$, $r(X_2)$, $\ldots$, $r(X_k)$, the list $r(X_k)$ has the biggest second-largest rank and denote it by $r(v_k)$, i.e., if $r(w_i)$ is the second-largest rank in $r(X_i)$ for some $i \in \{1, 2, \ldots, k-1\}$, then $r(v_k) \geq r(w_i)$. Let $r(u_k)$ be the largest rank in $r(X_k)$. Clearly, $r(u_k) \geq r(v_k)$. Note that the equality may hold.

Removing the list $r(X_k)$ from our consideration, we assume without loss of generality that among all $k-1$ lists $r(X_1)$, $r(X_2)$, $\ldots$, $r(X_{k-1})$, the list $r(X_{k-1})$ has the biggest third-largest rank and denote it by $r(v_{k-1})$. Let $r(u_{k-1})$ be the second-largest value of $r(X_{k-1})$. Clearly, $r(u_{k-1}) \geq r(v_{k-1})$. Since rank $r(v_k)$ is the biggest among all second-largest ranks in the lists, we have $r(u_{k-1}) \leq r(v_k)$. Since u_{k-1} and v_k have different neighborhoods, the strict inequality holds. As a result, we have $r(v_{k-1}) \leq r(u_{k-1}) < r(v_k) \leq r(u_k)$.

Suppose that we have picked i pairs ranks $r(u_k)$, $r(v_k)$, $\ldots$, $r(u_{k-i+1})$, $r(v_{k-i+1})$. Removing lists X_k, X_{k-1}, $\ldots$, X_{k-i+1} from our consideration, we assume without loss of generality that among the remaining $k-i$ lists $r(X_1)$, $r(X_2)$, $\ldots$, $r(X_{k-i})$, the list $r(X_{k-i})$ has the biggest $(i+2)$-th largest rank and denote it by $r(v_{k-i})$. Let $r(u_{k-i})$ be the $(i+1)$-th largest rank in $r(X_{k-i})$. According to our choices, we have $r(v_{k-i}) \leq r(u_{k-i}) < r(v_{k-i+1}) \leq r(u_{k-i+1})$.

Continuing in this fashion, we find $2k$ vertices u_1, v_1, $\ldots$, u_k, v_k such that for each $i \in \{1, 2, \ldots, k-1\}$, $r(v_i) \leq r(u_i) < r(v_{i+1}) \leq r(u_{i+1})$, which completes the proof of Claim 2. $\square$

Theorem 2.3. $\Theta(K^c_{n_1,n_2,\ldots,n_k}) \leq 2k-1$.

Proof. By Theorem 2.1, we have $\Theta(K_{n_1,n_2,\ldots,n_k}) \leq 2k-2$. Hence, by Theorem 1.1 we have $\Theta(K^c_{n_1,n_2,\ldots,n_k}) \leq \Theta(K_{n_1,n_2,\ldots,n_k}) + 1 \leq 2k-1$. $\square$

Note that $K^c_{n_1,n_2,\ldots,n_k}$ is a union of k disjoint cliques of orders n_1, n_2, $\ldots$, n_k. Although the proof of Theorem 2.4 on $K^c_{n_1,n_2,\ldots,n_k}$ is similar to the proof of Theorem 2.2, we give the proof for completeness.

Theorem 2.4. *If $n_i \geq k+1$ for each $i \in \{1, \ldots, k\}$, then $\Theta(K^c_{n_1,n_2,\ldots,n_k}) \geq 2k-1$.*

Proof. Let $G := K^c_{n_1,n_2,\ldots,n_k}$ and assume $\Theta(G) = t$. Let $\theta_1 < \theta_2 < \cdots < \theta_t$ and an assignment $r : V(G) \to \mathbb{R}$ be a t-threshold representation of G. Next, we show $t \geq 2k-1$.

Similar to the proof of Claim 2, we can show that there exist k pairs of vertices $u_i, v_i \in X_i$ with $i \in \{1, 2, \ldots, k\}$ such that $\max\{r(u_i), r(v_i)\} < \min\{r(u_j), r(v_j)\}$ whenever $i < j$. We assume without loss of generality that $r(u_i) \geq r(v_i)$ for each $i \in \{1, 2, \ldots, k\}$. Hence, for each $1 \leq i \leq k-1$, we have $r(v_i) \leq r(u_i) < r(v_{i+1}) \leq r(u_{i+1})$. We further acquire

$$r(u_i) + r(v_i) < r(u_i) + r(v_{i+1}) < r(u_{i+1}) + r(v_{i+1}). \tag{2}$$

Since $u_i v_i \in E(G)$ and $u_i v_{i+1} \notin E(G)$, there are odd number of thresholds among θ_1, θ_2, $\ldots$, θ_t between $r(u_i) + r(v_i)$ and $r(u_i) + r(v_{i+1})$, and so there is at least one threshold θ_{ℓ_i} such that $r(u_i) + r(v_i) < \theta_{\ell_i} \leq r(u_i) + r(v_{i+1})$. Similarly, we can show that there is at least one threshold θ_{m_i} such that $r(u_i) + r(v_{i+1}) < \theta_{m_i} \leq r(u_{i+1}) + r(v_{i+1})$. By (2), we have $\theta_{\ell_i} < \theta_{m_i}$.

In addition, $u_1 v_1 \in E(G)$, which in turn implies that there are odd number of thresholds among θ_1, θ_2, $\ldots$, θ_t less than or equal to $r(u_1) + r(v_1)$. And so there is at least one threshold θ_{ℓ_0} such that $\theta_{\ell_0} \leq r(u_1) + r(v_1)$.

Note that in general we have the following chain of inequalities

$$r(v_1) \leq r(u_1) < r(v_2) \leq r(u_2) < \cdots < r(v_k) \leq r(u_k).$$

Hence, we can obtain that

$$\theta_{\ell_0} < \theta_{\ell_1} < \theta_{m_1} < \theta_{\ell_2} < \theta_{m_2} < \cdots < \theta_{\ell_{k-1}} < \theta_{m_{k-1}}.$$

Consequently, there are at least $2k-1$ thresholds, and so $\Theta(G) \geq 2k-1$. $\square$

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